Semiclassical Decay of Near-Extremal Black Holes

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Abstract

Decay of a near-extremal black hole down to the extremal state is studied in the background field approximation to determine the fate of injected matter and Hawking pairs. By examining the behavior of light rays and solutions to the wave equation it is concluded that the singularity at the origin is irrelevant. Furthermore, there is most likely an instability of the event horizon arising from the accumulation of injected matter and Hawking partners there. The possible role of this instability in reconciling the D-brane and black hole pictures of the decay process is discussed.

1   Introduction

The decay of black holes by quantum emission of thermal radiation suggests that quantum gravity is not unitary[1]. Two types of information appear to be lost in the decay process, the state of the matter that falls into the black hole and the correlations between the Hawking quanta that are radiated away and their "partners" in the pair creation process. If the black hole evaporates completely this information seems to be destroyed, at least as far as the outside observer is concerned.

The argument that quantum gravity is not unitary is of course inconclusive at present since we lack a complete theory. According to one point of view the unitarity question hinges on the physics of the curvature singularity inside the black hole. The information that falls into the singularity might for example be destroyed, or leaked out to the exterior in a non-local process, or it might be passed on to a baby universe born at the singularity. Another viewpoint holds that it is not the singularity but the horizon that is the locus of hocus pocus.

The unitarity question has recently taken on a new light in view of a string theoretic correspondence between extremal (and near-extremal) black holes and certain D-brane configurations in perturbative string theory. Calculations of entropy and the rate of decay for these stringy states at weak coupling in flat spacetime agree spectacularly with totally different calculations for the corresponding black holes at strong coupling based on quantum field theory in a curved background black hole spacetime.

How far does the agreement between the string and black hole pictures go? Does the presence of the event horizon and singularity inside the black hole make any difference to the question of unitarity and the state of the outgoing radiation? In this paper the semiclassical description of the decay of a near-extremal black hole is studied as a first step in addressing these questions. The existence of a ground state makes this decay quite different from the evaporation of neutral (or discharging) black holes.

The model studied here is a spherically symmetric charged black hole that is excited above extremality and then allowed to decay back to the extremal ground state. This model was previously studied analytically by Strominger and Trivedi[2] in the large-N, adiabatic, S-wave approximation (two-dimensional reduction), including the back-reaction at one-loop order. Their work established the global structure of the spacetime in this approximation, from which they argued that an arbitrarily large amount of information can be injected into the black hole and lost to the outside world. The same model was also studied numerically, without the large-N or adiabatic approximations, by Lowe and O'Loughlin[3], whose results lend support to this picture, although of course they only evolved the system for a finite time. The focus of the present paper is in a sense complementary to the analyses of [2] and [3]. I do not implement a self-consistent model of the back-reaction; however I analyze more closely what is happening to the quantum fields inside the black hole.

This analysis suggests some rather surprising conclusions: First, the singularity at the origin is irrelevant. Second, the inner apparent horizon is probably quantum mechanically stable, unlike in the static case where the locally measured energy density grows exponentially with time. (If instead the inner horizon is unstable, then the semiclassical approximation breaks down, and information can fall into the strongly curved region.) If the inner horizon is indeed stable then the information in the black hole interior ends up sitting just behind the event horizon, and the *event* horizon is quantum mechanically unstable. It would be very interesting to see whether the numerical approach taken in [3] could be pushed to late enough times to check the picture of the instability arrived at here by adiabatic arguments. A horizon instability means of course that the semi-classical analysis is not as it stands self-consistent. In the Discussion section I will speculate on the role of this instability in reconciling the D-brane and black hole pictures, and in particular its relation to the singular horizons that occur in the spacetimes corresponding to D-brane configurations of non-maximal entropy.

2   The Model

The process I will consider is the following. Starting with an extremal black hole (which has vanishing Hawking temperature), with a stable charge[1], some matter is thrown in raising the temperature above zero. The black hole then emits Hawking radiation (in whatever quantum fields are present) and decays back to the extremal state. To model the spacetime of this process I use the charged Bonner-Vaidya metric [4, 5, 6] in four dimensions:

$$ds^2 = f(r,v)\,dv^2 - 2\,dvdr - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{1}$$

with

$$f(r,v) = 1 - \frac{2M(v)}{r} + \frac{Q(v)^2}{r^2}. \tag{2}$$

(Here and below I use units with $G = c = \hbar = 1$.) This line element is a solution to the Einstein-Maxwell-fluid equations with an ingoing charged null fluid. The charge density is proportional to $\dot{Q} \equiv dQ/dv$ and the energy-momentum tensor is $T_{\mu\nu} = \rho\nabla_\mu v\nabla_\nu v + T_{\mu\nu}^{\mathrm{em}}$, where $\rho = (4\pi r^2)^{-1}(\dot{M} - Q\dot{Q}/r)$ and $T_{\mu\nu}^{\mathrm{em}}$ is the electromagnetic field stress tensor for the radial electric field. If the mass M and charge Q are constant (1) is just the Reissner-Nordström black hole in ingoing Eddington-Finkelstein coordinates, and if $M = Q$ it is extremal. This is the non-compact part of one of the black holes used in the D-brane calculations[7, 8, 9, 10]. It seems not unlikely that the essential ideas discussed here would carry over to all near-extremal black holes. The Vaidya ($Q = 0$) and Vaidya-Bonner metrics have been used in many previous studies of evaporating black holes (see for example [11, 12, 2] and references therein).

   The Bonnor-Vaidya metric is not in detail the correct solution for our problem, since in the evaporation process the stress-energy tensor for the quantum fields is not described by a purely ingoing null flux of negative energy. Although an ingoing negative energy flux exists, there is also an outgoing positive energy flux outside the horizon, an outgoing negative energy flux inside, and vacuum polarization terms, However, if the evaporation process is adiabatic until a large stress energy develops, the behavior of the quantum fields on the black hole background should be reasonably well modeled by employing a sequence of Reissner-Nordström metrics with fixed charge and decreasing mass. The arguments of this paper depend only on the adiabaticity. In particular, the conclusion that a large stress energy develops somewhere is reliable in this approximation, although the subsequent evolution would need to be understood in a dynamically consistent manner.

   Now let us fix $Q(v) = Q \gg 1$, and define $\mu(v)$ as the mass above extremality,

$$M(v) = Q + \mu(v), \tag{3}$$

so that

$$f(r,v) = (1 - \frac{Q}{r})^2 - \frac{2\mu(v)}{r}. \tag{4}$$

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[1]In the string calculations the charge cannot be radiated away because only very massive solitons carry the charge. This is why the black hole decays back to an extremal state rather than discharging and evaporating completely.

When the black hole is absorbing mass $\mu(v)$ is increasing. When the Hawking radiation is emitted there is a negative energy flux into the black hole. To model this process I take $\mu(v)$ to be decreasing during that period. Thus $\mu(v)$ starts out zero, grows to a maximum, and then shrinks back to zero. The rate of decrease of μ is the luminosity, $\dot{\mu} \sim -T_H^4 A$, where $T_H = \kappa/2\pi$ is the Hawking temperature and κ is the surface gravity. For a static near-extremal black hole ($\mu \ll Q$) we have

$$\kappa = \frac{1}{2}(df/dr)|_{f=0} \simeq \sqrt{2\mu/Q^3}, \tag{5}$$

so $\dot{\mu} \sim -\mu^2/Q^4$, which implies

$$\mu(v) \sim Q^4/v. \tag{6}$$

That is, μ decays back to zero as v^{-1}, decreasing to half its initial value in a "half-life" of order Q^4/μ. Note that the Hawking temperature is much lower than μ until μ decreases to something of order $1/Q^3$, so that the semiclassical treatment of Hawking radiation looks quite reasonable for almost all of the decay process. Also, it is known[13] that the zero temperature static vacuum state for a massless scalar field with arbitrary curvature coupling is regular on the event horizon of an extremal four-dimensional black hole.[2] Thus the semiclassical treatment of the decay initially appears justified. We shall find later that a horizon instability at late times calls this into question however.

The zeroes of the metric component $f(r, v)$ are located at

$$r_{\pm} = Q + \mu \pm \sqrt{2\mu Q + \mu^2}. \tag{7}$$

For constant μ, r_+ is the event horizon and r_- is the inner horizon or Cauchy horizon inside the Reissner-Nordström black hole[15]. Between r_- and r_+ the function f, which is the norm of the "time" translation Killing field $\partial/\partial v$, is negative. This is the ergoregion. It is also the region of outer trapped surfaces. The ingoing radial light rays satisfy $dv = 0$, while the "outgoing" ones are given by $f\,dv = 2dr$. Thus where $f < 0$ the outgoing radial light rays are in fact going to smaller values of r. In the extremal case $\mu = 0$, the two horizons coincide, and there is no ergoregion. This is why there is no Hawking radiation: no negative energy states are available for the partners of the Hawking quanta.

In the dynamical case, where $\mu(v)$ grows and then shrinks back to zero, the two zeroes of f split and then come back together (see Fig. 1). The region where f is negative forms a blister on the extremal horizon containing trapped surfaces. I call the outer boundary of this region at r_+ the *outer apparent horizon* (or sometimes just *apparent horizon*), and the inner boundary at r_- the *inner apparent horizon* (or sometimes just *inner horizon*). The width in r of the trapped region at constant v is $\Delta r := r_+ - r_-$. In the near-extremal case (7) yields

$$\Delta r \simeq \sqrt{8\mu Q}. \tag{8}$$

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[2]Numerical calculations in [13] showed that the stress tensor is regular, but its derivatives were not examined. In the two-dimensional case[14] the stress tensor blows up on the horizon, but when one loop corrections are incorporated self-consistently in a dilaton-gravity-matter model, the divergence is postponed to the second derivative of the stress tensor.
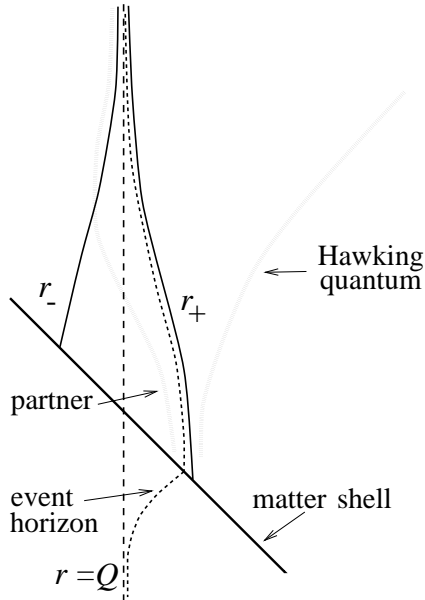
Figure 1: Decay of a near-extremal black hole in Eddington-Finkelstein coordinates (v, r). Lines of constant r are vertical and those of constant v slope at $45°$ towards the upper left. The curves $r_\pm(v)$ are the outer and inner apparent horizons. Hawking production of a pair of localized wavepackets is sketched.

When the event horizon enters the trapped region it begins shrinking monotonically as the black hole loses mass. It must remain inside the trapped region (where $f < 0$) from that point on until the trapped region goes away (unless more positive energy matter is thrown in at a later time). Note that at the boundary of the trapped region $f = 0$, so the outgoing light rays must be 'vertical' there, i.e., $dr/dv = 0$.

If the extremal black hole is excited by a macroscopic amount to a near-extremal state, then it will remain for a long time as an approximate Reissner-Nordström black hole, slowly decaying back to extremality. This is illustrated in Fig. 1, in which the injected mass is taken to be a null shell. Since the outgoing light rays are vertical wherever $f = 0$, while the inner and outer apparent horizons are moving out and in respectively, the inner horizon is evidently spacelike while the outer (apparent) horizon is timelike. The event horizon remains very close to the apparent horizon throughout the decay process. One can estimate just how close by noting that the event horizon must stay outside $r = Q$ for a time of order the initial half-life Q^4/μ. Using the expression given in the next section for the time for a light ray to peel away from near the horizon of a static black hole, this implies that the initial radial coordinate of the event horizon must be $r_+ - \epsilon$, with

$$\epsilon \sim \sqrt{\mu Q} \exp(-\sqrt{Q^5/\mu}), \tag{9}$$

a very small distance indeed.

3   Decay of a near-extremal black hole

The essential point of this paper arises from the elementary observation that the behavior of "outgoing" light rays inside the black hole is qualitatively very different for an extremal black hole than it is for a neutral or charged nonextremal one. The outgoing radial light rays are obtained by integrating the equation

$$dr/dv = f/2. \tag{10}$$

In the neutral (Schwarzschild) black hole, these rays peel away from the event horizon and fall to the (spacelike) singularity at $r = 0$ in a time (with respect to the ingoing advanced time coordinate v) of order $r_+ \ln(r_+/\epsilon)$, where $r_+ - \epsilon$ is the initial radial coordinate. In the non-extremal charged black hole, these rays peel away from the event horizon reaching the midpoint $(r_+ + r_-)/2$ in a time of order $[r_+^2/(r_+ - r_-)] \ln[(r_+ - r_-)/\epsilon]$. Then, rather than falling into the (timelike) singularity at $r = 0$, they asymptotically approach the inner horizon in a time that diverges as $[r_-^2/(r_+ - r_-)] \ln[(r_+ - r_-)/\epsilon]$, where now $r_- + \epsilon$ is the final radial coordinate. In the extremal case, on the other hand, the outgoing light rays *never* peel away from the event horizon. Instead they *approach* the event horizon as $(r_+ - r) \sim Q^2/v$.

We should pause here to understand exactly in what sense the outgoing light rays get "close to the horizon". An invariant description can be given by reference to the freely falling observers that start at rest at infinity and fall across the horizon. As $v \to \infty$, the outgoing light ray inside the event horizon will be intercepted by these observers an arbitrarily short proper time after they cross the horizon.[3] It is worth pointing out that this is somewhat counter-intuitive from the point of view of the Penrose diagram for an extreme Reissner-Nordström metric, which is depicted in Fig. 2. Both the inner horizon and the event horizon occur at the radial coordinate Q. The Eddington-Finkelstein coordinates cover only the interior of the trapezoidal region ABi^0i^-, so the inner horizon is not included in the Eddington-Finkelstein patch. The outgoing light ray γ actually crosses the inner horizon (in a finite affine parameter, although the advanced time goes to infinity there). Nevertheless, this ray also gets arbitrarily close to the event horizon, in the sense of the proper time of free-fall observers described above, even though on the Penrose diagram it can appear to cross the inner horizon very far from the event horizon.[4]

The difference in the behavior of the outgoing light rays inside the black hole is crucial for the fates of both the matter that excites the extremal black hole and the partners of the Hawking radiation. Basically it means that all the information ends up just inside the event horizon. To explain the idea let me first consider the Hawking process in a cartoon using the geometric optics approximation and purely radial light rays (cf. Fig. 1). Later I discuss

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[3]This follows from the radial equation (11). With $L = 0$, $E = 1$, and $\eta = 1$ (11) yields $\dot{r} = -(1 - f)^{1/2}$. Thus near the horizon one has $\dot{r} \simeq 1$, so that the radial coordinate passes at the same rate as the proper time.

[4]Similar comments can be made about the non-extremal case as well. There, the outgoing rays cross the "top right" part of the inner horizon at $r = r_-$ in a finite affine parameter, but they also approach arbitrarily close to the "top left" portion of the inner horizon (which is also at $r = r_-$ and *is* included in the Eddington-Finkelstein patch) in the sense described above.
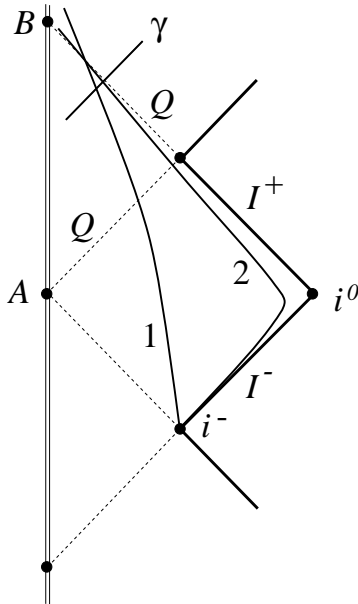
Figure 2: Penrose diagram of part of the maximally extended extremal Reissner-Nordström space-time. The interior of the trapezoid ABi^0i^- is the patch covered by Eddington-Finkelstein coordinates. The dotted lines labeled Q are the event horizon and inner horizon. The two curves labeled 1 and 2 are radial free-fall trajectories that begin at rest at infinity and fall into the black hole at different times. The proper time along the world line 2 between the event horizon and the crossing of the outgoing light ray γ goes to zero as the infall time of 2 goes to infinity.

the modifications of this picture brought in by angular momentum and wave behavior. These modifications do not change the essence of the picture for the Hawking radiation, but they are required to understand how the information in the injected mass reapproaches the horizon from the inside.

3.1   Fate of the Hawking partners

The pair to be created in the Hawking process straddles the event horizon[5]; equivalently, since the event horizon and apparent horizon are so close together (cf. (9)), the pair straddles the apparent horizon. While the Hawking quantum on the outside escapes, the partner falls across the trapped region and asymptotically approaches the inner horizon. As the black hole decays the inner horizon moves outward toward $r = Q$ on a *spacelike* trajectory as discussed in section 2. The partner must stay on a null ray, so it *crosses* the inner horizon into the interior region with $f > 0$. At this point it begins moving outward and asymptotically approaches the surface $r = Q$.

Evidently, then, all the partners of the Hawking radiation pile up just inside the event

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[5]The validity of this localized pair creation picture of the Hawking process has been elucidated by the work of and Parentani and Brout[16].

horizon, rather then falling into the black hole. This means that the information in their correlations with the Hawking quanta is available just across the horizon. For a neutral evaporating black hole, one can always choose to foliate the spacetime with surfaces that dip way back to the past in order to intercept all the partners before they reach the singularity. However this is an extremely distorted surface and, more importantly, once the black hole evaporates completely this possibility is no longer available. By contrast, in the near-extremal case, there is no need to dip the surface at all, and the information can be accessed in this way at any time, even long after the hole has decayed entirely back to extremality.

3.2   Horizon instability

Now let us consider more closely the role of the back reaction in the decay process. The partners carry negative energy. According to the preceding discussion, this energy piles up with ever increasing density just inside the event horizon. When the back-reaction is accounted for, this leads to some kind of instability of the horizon.

In arriving at this conclusion I have assumed that the partners do not encounter a region of large stress energy (where the back-reaction would be large and the adiabatic approximation would break down) *before* returning to the event horizon. However there seems to be a chance that exactly this would happen as the inner apparent horizon is approached. In a static, non-extremal Reissner-Nordström metric, the negative energy partners are stuck in the ergoregion between the two horizons and they pile up just outside the inner apparent horizon. In the Unruh vacuum, an observer who freely falls across this horizon at late times sees a very large outgoing negative energy density which grows exponentially with the time at which the observer falls across the inner apparent horizon.[6] Although this growing flux is purely outgoing, there is also presumably some finite ingoing flux, and together these mean that the invariant square of the stress tensor is getting large so there should be a large back-reaction. Might the same instability occur near the inner horizon of the *decaying* near-extremal black hole? After all, this black hole remains non-extremal for a very long time of order Q^4/μ (6), so perhaps there is a buildup of negative energy just outside the inner horizon in this case as well. If so, then rather than crossing the inner horizon and finally returning to the event horizon, the partners may encounter a (spacelike or null) singularity and never make it across the inner horizon and back out to the event horizon.

It seems however that such an instability of the inner horizon probably does *not* occur. The negative energy drives the inner horizon out on a spacelike trajectory towards the event horizon, which allows the negative energy to slip across before piling up too much. I have made a rough estimate of the amount of negative energy density that builds up, and it turns out to be of order $\mu/Q(\ln Q)^2$ which is very small (compared to Planck density) for the near-extremal black holes we are considering. The calculations behind this estimate are

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[6]Contrary to what I said in an earlier draft of this paper, this is *not* the same as the quantum instability at the *Cauchy* horizon of stationary non-extremal charged two-dimensional[17], four-dimensional[19], and charged rotating four-dimensional [18] black holes. On the Cauchy horizon the stress-energy tensor is actually *infinite*, and an observer who falls across the Cauchy horizon sees this divergence.

given in the Appendix of this paper. Though not rigorous, this argument at least makes the assumption the inner horizon is *not* unstable fairly plausible. It would be very interesting to check the validity of this assumption in a self-consistent numerical calculation like that of Ref. [3]. In the discussion section I will return to the instability issue.[7]

4 Angular momentum and wave behavior

The restriction to radial light rays was convenient in the preceding discussion. However, both the injected matter and the Hawking radiation may carry angular momentum, and they may scatter as waves, so it is necessary to see whether the basic picture described in the previous section survives when these effects are included. The Hawking radiation in anything other than the S-wave is suppressed by a factor $(\omega Q)^l$ which is a very small number of order $(\mu/Q)^{l/2} \ll 1$ for ω of order the Hawking temperature (cf. the surface gravity (5)). Nevertheless, it is still important in principle to determine the fate of the partners with $l \neq 0$, and moreover the injected matter need not have $l = 0$.

Let me begin by sticking with the geometric optics approximation. The conclusion will be that, except for the ingoing radial null geodesics, all the timelike or null geodesics miss the singularity and swing back out to asymptotically approach the event horizon. I will then extend the analysis to wave propagation, arriving at two conclusions. First, for the negative energy wavepackets inside the horizon there is very little scattering, and second, there is a reasonable way to handle the singularity, for which waves simply scatter back to large radius without being swallowed.

4.1 Geodesics with angular momentum

The geodesics in the line element (1) with constant Q and M may be taken to lie in the plane $\theta = \pi/2$. The remaining coordinates are functions of an affine parameter λ (which we take to be the proper time in the timelike case), $(v(\lambda), r(\lambda), \phi(\lambda))$. Defining the conserved energy $E = g_{v\alpha}\dot{x}^\alpha = f\dot{v} - \dot{r}$ and angular momentum $L = g_{\phi\alpha}\dot{x}^\alpha = r^2\dot{\phi}$, the equation $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \eta$ ($\eta = 0$ for null curves and $\eta = 1$ for timelike curves) becomes an equation for the radial coordinate:

$$\dot{r}^2 + U(r) = E^2 \tag{11}$$

with the effective potential $U(r)$ given by

$$U(r) = f(r)(L^2/r^2 + \eta). \tag{12}$$

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[7]A study of the stability of the interior of an evaporating Reissner-Nordström black hole was made by Kaminaga[12], who used a two-dimensional model so that the stress tensor could be computed exactly. In this work it was assumed that both the mass and the charge evaporate, with a fixed ratio, and the stress tensor was computed on this evaporating background. No attempt was made to enforce self consistency in the sense of (say) the four-dimensional semi-classical Einstein equation. One of Kaminaga's results was that the stress tensor blows up at the *Cauchy* horizon that forms. Unfortunately this study is of no direct help to us, since we are interested in decay with the charge fixed. This makes a big difference, because for us the inner horizon moves *out* whereas in the model of [12] it moves *in*. Also, the lack of self-consistency could well be important to the question of stability.

Near $r = 0$ we have $f(r) \simeq Q^2/r^2$, so there is a tremendous barrier unless both L and η vanish. That is, only radial null geodesics can reach the timelike singularity. Any other geodesic with $\dot{r} < 0$ inside the horizon reaches a minimum value of r and then proceeds to increasing values of r. The advanced time v goes to infinity when the inner horizon is crossed with $\dot{r} > 0$ for positive energy orbits and with $\dot{r} < 0$ for negative energy orbits (since $\dot{v} = (E + \dot{r})/f$). In the extremal case, for which $f(r) = (1 - Q/r)^2$, there are no negative energy orbits, and the geodesics all return to $r = Q$, crossing the inner horizon at a finite affine parameter as described previously for the radial, massless case. As in the radial case, these geodesics also come arbitrarily close to the event horizon.

The actual situation in the case of the decaying near-extremal black hole is of course time-dependent, but we can understand the nature of the trajectories there by considering an initial segment, propagating in a static, near-extremal, Reissner-Nordström metric, followed by a second segment beginning after the trajectory reaches its minimum value of r.

On the initial segment of a positive energy orbit (which is the relevant type of orbit for the infalling matter), the inner horizon at r_- is crossed after a finite affine parameter and a finite advanced time v. After reaching its minimum value of r at finite v the geodesic begins to move to larger r, reaching r_- again at finite affine parameter but infinite v. That is, the geodesic asymptotically approaches r_- from the inside. In the decaying case, the inner horizon (the inner zero of f) gradually moves outward toward the event horizon. Our geodesic follows it out, just as in the purely radial case described earlier.

The negative energy orbits (which are the relevant ones for the Hawking partners) behave a bit differently. In the static phase they take an infinite advanced time to cross the inner horizon on the way in. In the decaying case, they slip across the inner horizon after a finite advanced time. At the inner horizon they have $\dot{r} = -(L^2/r^2 + \eta)/2\dot{v} < 0$. Since they spend a long time near the almost static inner horizon, this negative $\dot{r}$ must be very small. The orbits thus continue in a little bit to some minimum radius, and then move out again following the inner horizon to the event horizon. Thus, in fact, allowing for angular momentum changes nothing essential about the process.

4.2   Wave propagation

The geometrical optics analysis suggests that if the quantum field is treated properly using a wave equation, the singularity may be avoided and the waves will scatter back out to the horizon. To study this question, let us consider for example matter satisfying the massless Klein-Gordon equation. The scattering can cause dispersion of wavepackets, so the simple picture of where the Hawking partners go may need revision. Even the outgoing S-wave just inside the horizon can in principle backscatter from the curvature and fall in towards the origin. More fundamentally, the timelike singularity at $r = 0$ must be tamed in some fashion in order to make sense of the wave equation. This amounts to the question of boundary conditions at the origin.

To understand the basic physics, it should be adequate to consider only the static near-extremal Reissner-Nordström spacetime. The wave equation can be separated in both

Eddington-Finkelstein type coordinates (1) and diagonal coordinates

$$ds^2 = f(-dt^2 + dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{13}$$

where

$$f(r) = (r - r_-)(r - r_+)/r^2 \tag{14}$$

and the tortoise coordinate r_* is related to r by

$$dr_* = f^{-1}dr. \tag{15}$$

The diagonal coordinates cover the entire interior up to $v = \infty$ in the extremal case, but they are singular on the inner horizon in the non-extremal case. Thus one can use them to discuss either the region between the two horizons or the region inside the inner horizon. (To discuss both regions together the Eddington-Finkelstein coordinates would be preferred.)

Writing the matter field Φ as

$$\Phi = \frac{u(r)}{r} Y_{lm}(\theta, \phi)e^{-i\omega t} \tag{16}$$

the wave equation reduces to the radial equation

$$-\frac{d^2}{dr_*^2}u = [\omega^2 - V(r)]\,u \tag{17}$$

with

$$V(r) = f\left[f'/r + l(l+1)/r^2\right] \tag{18}$$

where $f' = df/dr$. Note that the second term in $V(r)$ is similar to the effective potential (12) for massless geodesic motion, but there is no geodesic analog for the f' term.

The geometrical optics limit of the negative energy wavepackets is described by negative energy light rays. Let us follow the trajectory of these light rays and estimate the amount of scattering suffered by a wavepacket. The rays that begin just inside the horizon r_+ sink down to r_- asymptotically approaching the inner horizon as discussed in the previous subsection. In between r_- and r_+ the maximum value of $|ff'|/r$ is bounded by

$$|ff'|/r < |f_{\max}f'(r_-)|/r_- = (r_+ - r_-)^3/4r_+r_-^4 \sim \mu^{3/2}Q^{-7/2}. \tag{19}$$

For S-waves we therefore have $\omega^2/|V| \gtrsim (\omega/T_H)^2(Q/\mu)^{1/2}$, so there is extremely little scattering except possibly for very low frequencies compared to the Hawking temperature.

For nonzero angular momentum, although the potential term in (17) is important, there is still very little wave scattering, since the wavelength remains small compared to the radius of curvature Q of the spacetime. To see this note that, when $l \neq 0$, $V(r)$ is negative everywhere between r_- and r_+. Thus in the eikonal approximation we have $k_* > \omega$. Since $k_* = fk$ (15), this yields

$$k > \omega/f > \omega/|f|_{\max} \gtrsim (\omega/T_H)(\mu Q)^{-1/2} \tag{20}$$

(which holds also for S-waves). Thus, although the wavelength does not remain small compared with $r_+ - r_- \sim \sqrt{\mu Q}$, it certainly does remain small compared to the radius of curvature Q except for very low frequencies $\omega \lesssim (\mu/Q)^{1/2} T_H$.

The negative energy wavepackets are thus well described by the geometric optics approximation when they cross the inner horizon, and it is clear the approximation will remain good the rest of the way. As the metric approaches the extremal one, the trajectory proceeds outward towards the event horizon, always remaining at small values of f.

The positive energy wavepackets on the other hand will clearly fall deep into the black hole and scatter. In the geometrical optics approximation we saw that trajectories with angular momentum will dip inside r_- before coming up again. I will not attempt here to estimate the amount deviation from geometric optics, but rather ask what is the general nature of scattering close to the origin. As long as these modes do not get swallowed or trapped by the singularity, it is quite plausible that they too eventually make it back out to the event horizon. Thus let us next consider scattering deep inside the black hole near the singularity.

To this end note that the most singular term in (18) is not the centrifugal barrier but the f'-term, which diverges as $-2Q^2/r^3$ near $r = 0$. Meanwhile f diverges as Q^2/r^2, so we have

$$V(r) \simeq -\frac{2Q^4}{r^6} \qquad \text{as} \quad r \to 0. \tag{21}$$

To use the radial equation (17) this should be expressed in terms of $r_* = \int f^{-1} dr \simeq r^3/3Q^2$, yielding

$$V(r_*) \simeq -\frac{2}{9} \frac{1}{r_*^2}. \tag{22}$$

The Schrödinger-like equation (17) in an attractive potential $-\gamma/r_*^2$ has solutions near the origin of the form r_*^s, where $s = (1 \pm \sqrt{1-4\gamma})/2$. If $\gamma > 1/4$ the solutions oscillate an infinite number of times as the origin is approached. In our problem $\gamma = 2/9$, so there are no oscillations and we have $s = 2/3$ and $s = 1/3$. The behavior of $u(r)/r$ for these two solutions is r^1 and r^0. Thus there is no "singular" behavior at the singularity. On the other hand, since both solutions are square integrable in the appropriate measure[8], a boundary condition must be supplied to select a unique solution. This boundary condition is presumably supplied by the physics at the singularity. The phase shift between incoming and outgoing waves depends on this boundary condition, but the fact that the ingoing partial waves emerge as outgoing waves and return to large values of r does *not*. For our purposes this is enough.[9]

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[8]This is the measure for which the spatial differential operator in the wave equation is symmetric[20], i.e., $f^{-1/2}dv$, where dv is the proper volume element.

[9]The existence of the two regular solutions means that the Hamiltonian is not "essentially self-adjoint". In [20] Horowitz and Marolf investigated when the Hamiltonian for a Klein-Gordon field remains essentially self-adjoint in the presence of timelike singularities. They mention the result found here for the Reissner-Norström black hole, but also identify other black hole singularities for which essential self-adjointness does hold.

5 Discussion

Now that we have some insight into the semiclassical decay of near-extremal black holes let us compare it with the corresponding D-brane decay. A natural process to consider is the excitation of an extremal configuration followed by decay back to the extremal state. A complication arises, however, in the D-brane description of this process. Even if the extremal D-brane configuration is in the maximal entropy state (the microcanonical ensemble for the fixed set of charges) to begin with (which is the usual assumption in D-brane calculations[10]), after being excited its state will depend in a complicated way on the state of the absorbed quanta and its interaction with them. Since the agreement found between D-brane and black hole entropies and radiation rates holds when the D-brane state has maximal entropy, the two processes would already differ immediately after the energy absorption. Thus instead let us consider a process that starts off with an already excited near-extremal D-brane state with maximal entropy, and compare this to the semiclassical decay of a near-extremal black hole.

The string dynamics is unitary, so the final state has the same entropy as the initial one. This state is correlated in the D-brane and radiation degrees of freedom. The reduced density matrix of the D-brane configuration is the maximal entropy state (since the original state before decay was the microcanonical ensemble), and the reduced density matrix of the radiation is that of thermal radiation from a "greybody". The semiclassical decay is also unitary in the standard sense of quantum field theory in curved spacetime if the background is treated classically and if the horizon instability is ignored. Then the state of the quantum field itself remains pure if it started out pure, with the Hawking radiation correlated to the state of the field inside the horizon.[11]

Although the spacelike surfaces that foliate the extremal black hole spacetime include the timelike singularity at the origin (which has no counterpart in the D-brane configuration), we have seen that the Hawking partners essentially never go near the singularity but rather end up hovering just inside the event horizon, and the waves that do scatter deep into the black hole just scatter back out again and approach the horizon (although the phase shift upon scattering through the origin depends on an unknown boundary condition there). Thus the singularity at $r = 0$ appears to be irrelevant to the unitarity question in this case.

Neglecting the horizon instability is, however, inconsistent in the semiclassical approach, in which the black hole background decays in response to quantum field energy. What is the nature of this instability? I do not know, but I will offer some speculative remarks. If the inner apparent horizon is indeed stable, as argued in the Appendix, then there is certainly pile up of negative energy Hawking partners (as well any previously injected positive energy[12]) just behind the inner apparent horizon which at late times is just inside the event

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[10]The mixed character of these D-brane states was recently emphasized by Myers[21].

[11]Allowing the background spacetime to fluctuate would go beyond the semiclassical calculation, and would presumably also preserve purity of the joint state.

[12]The positive energy of the injected matter should be balanced by the total negative energy flux into the black hole, and if these two energies were identically distributed, one might expect them to cancel, leaving the horizon stable. However there is no reason why the injected matter and the Hawking radiation partners

horizon. This is a purely null flux, however, so it will yield a large invariant such as $T_{\mu\nu}T^{\mu\nu}$ only if there is also a finite ingoing flux. There is of course the ingoing negative energy flux associated with the Hawking radiation, but this is going to zero at late times. I do not know what the late time limit of this invariant is. A large invariant will certainly develop if more matter is injected to re-excite the black hole above extremality. In this case, the instability would be located just inside where the event horizon would have been had the new matter not been injected. The actual event horizon, on the other hand, moves out in anticipation of the arrival of the injected matter, so if enough matter is injected the event horizon will be far outside this instability!

For the purposes of comparison with the D-brane picture, the most important question about the buildup of energy inside the horizon is whether or not it affects the subsequent Hawking radiation in any way. If it were located where the Hawking pairs are born, it would certainly have an impact. However, after the extremal hole is re-excited, the event horizon has moved away from the region of large energy density, and when they are born the new Hawking pairs do not see this large energy density. As usual, to determine the pair creation amplitude one must follow the pair backwards in time to see if it emerges from a vacuum state. In this case, even though the pair gets very close to the inner apparent horizon during the quiescent period when the hole is extremal, the pair never gets *inside* the apparent horizon, so even in its past it never seems to see the large energy density. This naive argument thus suggests that the subsequent Hawking radiation is insensitive to presence of the large energy density provided the event horizon itself is indeed not singular.

However, even if the event horizon is not *necessarily* singular after the near-extremal black hole initially decays, the huge null negative energy flux behind the horizon is bizarre and makes the situation highly unstable, since the slightest influx would be catastrophic. It seems at first that such an unstable event horizon would ruin the agreement with the D-brane picture, but in fact there is also evidence from the theory of D-brane decay that a horizon instability occurs. When a near-extremal, nonsingular (in the sense of the strong coupling analog) D-brane state decays, the final state is a mixed and the Hawking radiation is correlated to the microstate of the D-brane configuration. In any particular realization of the Hawking radiation, a partial projection onto a sub-ensemble with a *non-uniform* distribution of charge presumably occurs. Small changes of the internal charge distribution of a D-brane state whose strong coupling analog is a black hole (or string) with nonsingular horizon seem to generally correspond to a black holes with a singular "would-be" horizon[22, 23, 24]. It therefore appears that the D-brane configuration evolves into a state whose strong coupling analog has a singular horizon.[13]

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should end up equally distributed in general. Even if the injected matter is purely an S-wave, when it arrives back at the horizon it will be located at slightly smaller radius than all the Hawking partners (which are also primarily S-waves) inside the event horizon. This would produce a "dipole" structure in the energy density.

[13]A partial projection would also occur in the *pure* state semi-classical description of the Hawking radiation. In that case Massar and Parentani [25] showed that a quantity that may be called the conditional expectation value of the stress tensor is huge and oscillating near the horizon. However, in this pure state case, blurring the "post-selection" of the state can presumably average over the wild fluctuations leaving a small residual. In the mixed state D-brane case, *every* member of the ensemble is "singular", and one of them is in principle

A horizon instability would mean a failure of the semiclassical approximation, which may be just what is needed to avoid a discrepancy with the D-brane picture over the question of entropy of the reduced state of the radiation when a black hole or D-brane configuration is repeatedly excited and allowed to decay. In the D-brane case, the entropy of the final D-brane configuration itself is bounded by the maximal entropy of the microcanonical ensemble. Since the whole process is unitary, the entropy of the radiation cannot just be that of so much thermal radiation. Instead, there must be correlations in the radiation emitted at different times. In the black hole case, if the horizon remains regular, Hawking's analysis requires that the radiation be purely thermal. This would be consistent with unitary evolution in the semiclassical framework once the degrees of freedom of the quantum field inside the black hole are accounted for. However, the reduced state of the Hawking radiation would have a much larger entropy than that of the D-brane radiation. If instead the horizon is unstable after the initial decay process, then perhaps the repeated excitation of the black hole cannot be described in the semiclassical approximation.

The preceding discussion seems at first to offer a possible resolution of how it could be that the D-brane and semiclassical black hole calculations agree so well with respect to the rate of radiation, yet disagree as to the long term evolution—although we have to destroy the black hole in order to save it. However, it seems this cannot be the whole answer, since a small change in the process leads to trouble. Instead of repeatedly exciting the system and letting it decay, one might send in a constant flow of positive energy matter to maintain the system indefinitely in a near-extremal state. In this case the Hawking partners pile up at the inner horizon and stay there, well away from the event horizon. It is the inner horizon that is then unstable, but this does not influence the Hawking radiation. Disagreement with the D-brane picture over the state of the radiation then seems unavoidable. Another example of a process where the pictures disagree was disussed by Maldacena and Strominger[26]. They pointed out that one can find a regime in which it is possible to excite an extremal black hole to a near-extremal configuration whose corresponding black hole entropy is much greater than the entropy of the original extremal configuration. In such a process the resulting Hawking radiation would have a much larger entropy than the corresponding D-brane could radiate.

In view of these examples, it is clear that something is still missing in our understanding of the uncanny (partial) agreement between the weakly coupled D-brane model and the strongly coupled black hole model. Hopefully the semiclassical study presented here can be helpful in addressing this question.

Acknowledgments

actual.

Appendix: Energy density at the inner apparent horizon

For a static Reissner-Nordström black hole the inner (apparent) horizon is an infinite blue shift surface. Between the inner and outer horizons an outgoing light ray falls inward and asymptotically approaches the inner horizon. An outgoing null flux of (negative) energy between the horizons will thus pile up just outside the inner horizon. In the static case, the energy density observed on a free-fall world line crossing the inner horizon will grow exponentially like

$$\rho \sim e^{2\kappa v} \rho_0, \tag{23}$$

where κ is the surface gravity of the inner horizon, v is the advanced time, and ρ_0 is some initial energy density. This quickly exceeds the Planck density so, as long as there is a finite ingoing flux as well, the square of the stress tensor will be a large invariant and the back-reaction will be large.

In the evaporating case the inner horizon is moving out, allowing the energy to slip across. Here we estimate how large the energy density gets near the inner horizon. We do this by estimating how much advanced time passes before a given outgoing light ray crosses the inner horizon.

In the static near-extremal case, an outgoing light ray it falls from the midpoint $(r_+ + r_-)/2$ to the position $r_- + x$ in an advanced time v given by

$$x \sim (r_+ - r_-) \, e^{-\kappa v} \sim \kappa Q^2 \, e^{-\kappa v}, \tag{24}$$

where κ is the surface gravity (5) which is approximately the same for both horizons. If the hole is evaporating slowly this should still hold to a good approximation. Meanwhile the inner horizon is moving out. At the inner horizon $f(v, r_-(v)) = 0$, so $\dot{r}_- = -f_v/f_r \simeq -\dot{\mu}/\kappa Q \sim \kappa^3 Q$. The change Δr_- in the radius of the inner horizon over a time v is approximately $\dot{r}_- v$, from which we find

$$\Delta r_- \simeq \kappa^3 Q v. \tag{25}$$

The time at which the outgoing light ray meets the inner horizon is now found by setting $\Delta r_- = x$, which yields

$$e^{\kappa v} \sim Q/\kappa^2 v. \tag{26}$$

Since $Q/\kappa \gg 1$, we find

$$e^{\kappa v} \sim Q/\kappa \ln Q. \tag{27}$$

Using this and choosing the initial density $\rho_0 \sim \kappa^4$ (which seems reasonable for the Unruh vacuum) we obtain from (23)

$$\rho_{\max} \sim \frac{\mu}{Q(\ln Q)^2} \ll 1. \tag{28}$$

By this estimate the energy density stays very small at the inner horizon of an evaporating near-extremal black hole.

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